

A HEAT EQUATION ON A QUATERNIONIC CONTACT MANIFOLD

S. IVANOV AND A. PETKOV

ABSTRACT. A quaternionic contact (qc) heat equation and the corresponding qc energy functional are introduced. It is shown that the qc energy functional is monotone non-increasing along the qc heat equation on a compact qc manifold provided certain positivity conditions are satisfied.

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1. INTRODUCTION

We introduce a quaternionic contact (qc) heat equation and the corresponding qc energy functional. The purpose of this paper is to show that the qc energy functional is monotone non-increasing along the qc heat equation on a compact qc manifold provided certain positivity conditions are satisfied. In dimensions at least eleven the positivity condition coincides with the Lichnerowicz-type positivity condition used in [5, 6] to derive a sharp lower bound for the first eigenvalue of the sub-Laplacian on a compact qc manifold. In dimension seven, in addition, we need to assume the positivity of the introduced in [7] P-function.

It is well known that the sphere at infinity of a non-compact symmetric space of rank one carries a natural Carnot-Carathéodory structure, see [10, 11]. A quaternionic contact (qc) structure, [1], appears naturally as the conformal boundary at infinity of the quaternionic

Date: August 2, 2016.

2010 *Mathematics Subject Classification.* 53C21, 58J60, 53C17, 35P15, 53C25.

Key words and phrases. quaternionic contact structures, heat equation, energy functional, Lichnerowicz inequality, P-function.

hyperbolic space. Following Biquard, a quaternionic contact structure (*qc structure*) on a real $(4n+3)$ -dimensional manifold M is a codimension three distribution H (*the horizontal distribution*) locally given as the kernel of a \mathbb{R}^3 -valued one-form $\eta = (\eta_1, \eta_2, \eta_3)$, such that the three two-forms $d\eta_i|_H$ are the fundamental forms of a quaternionic Hermitian structure on H . In other words, a quaternionic contact (qc) manifold (M, g, \mathbb{Q}) is a $4n+3$ -dimensional manifold M with a codimension three distribution H equipped with an $Sp(n)Sp(1)$ structure. Explicitly, H is the kernel of a local 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in \mathbb{R}^3 together with a compatible Riemannian metric g and a rank-three bundle \mathbb{Q} consisting of endomorphisms of H locally generated by three almost complex structures I_1, I_2, I_3 on H satisfying the identities of the imaginary unit quaternions.

On a qc manifold one can associate a linear connection with torsion preserving the qc structure, see [1], which is called the Biquard connection. One defines the horizontal Ricci-type tensor with the trace of the curvature of the Biquard connection, called the qc Ricci tensor. This is a symmetric tensor [1] whose trace-free part is determined by the torsion endomorphism of the Biquard connection [4] while the trace part is determined by the scalar curvature of the qc-Ricci tensor, called the qc-scalar curvature.

Let (M, g, \mathbb{Q}) be a compact qc manifold. We consider *the qc heat equation*

$$(1.1) \quad \frac{\partial}{\partial t} u = -\Delta u,$$

where $u(x, t) : M \times [0, +\infty) \rightarrow \mathbb{R}$ is smooth function and $\Delta : \mathcal{F}(M) \rightarrow \mathcal{F}(M)$ is the sub-Laplacian on M . From now on, u will be a positive solution of (1.1). We introduce the functions $\varphi \stackrel{\text{def}}{=} -\ln u$ and $F \stackrel{\text{def}}{=} u^\alpha$, where $\alpha \in \mathbb{R}, \alpha \neq 0, \frac{1}{2}$. The energy functional for (1.1) is defined by

$$(1.2) \quad \mathcal{F}(\varphi) = \int_M |\nabla \varphi|^2 e^{-\varphi} \text{Vol}_\eta.$$

Our main result follows.

Theorem 1.1. *Let (M, g, \mathbb{Q}) be a compact $4n+3$ -dimensional quaternionic contact manifold and the Lichnerowicz type condition (1.3) holds, $L(X, X) \geq 0$ for any $X \in \Gamma(H)$.*

- i) *If $n > 1$ then the energy functional (1.2) is monotone non-increasing along the qc heat equation (1.1).*
- ii) *In the case $n = 1$ suppose in addition that the P -function of any $F^{\frac{1}{2\alpha}}$, corresponding to a (positive) solution u of (1.1) is non-negative. Then the energy functional (1.2) is monotone non-increasing along the qc heat equation (1.1).*

The Lichnerowicz type assumption cf. (2.2), (2.6),

$$(1.3) \quad L(X, X) = \text{Ric}(X, X) + \frac{2(4n+5)}{2n+1} T^0(X, X) + \frac{6(2n^2+5n-1)}{(n-1)(2n+1)} U(X, X) \\ = 2(n+2)Sg(X, X) + \frac{4n^2+14n+12}{2n+1} T^0(X, X) + \frac{4(n+2)^2(2n-1)}{(n-1)(2n+1)} U(X, X) \geq k_0 g(X, X),$$

(the third term in the left-hand side is dropped if $n = 1$) yields a sharp lower bound of the first eigenvalue of the sub-Laplacian when $n > 1$ [5] while for $n = 1$ one needs additional assumption expressed in terms of the positivity of the P -function defined in [6] to achieve the

validity of the same lower bound [6]. The P -function of a smooth function f is defined with the help of the Biquard connection, the qc-scalar curvature and the $Sp(n)Sp(1)$ -components of the torsion tensor see (2.8) below.

Convention 1.2.

- a) We shall use X, Y, Z, U to denote horizontal vector fields, i.e. $X, Y, Z, U \in H$.
- b) $\{e_1, \dots, e_{4n}\}$ denotes a local orthonormal basis of the horizontal space H .
- c) The triple (i, j, k) denotes any cyclic permutation of $(1, 2, 3)$.
- d) s will be any number from the set $\{1, 2, 3\}$, $s \in \{1, 2, 3\}$.

Acknowledgments The authors thank Dimiter Vassilev for stimulating conversations. The research is partially supported by Contract DFNI I02/4/12.12.2014 and by the Contract 195/2016 with the University of Sofia ‘St.Kl.Ohridski’.

2. QUATERNIONIC CONTACT MANIFOLDS

Quaternionic contact manifolds were introduced in [1]. We also refer to [4] and [8] for further results and background.

2.1. Quaternionic contact structures and the Biquard connection. A quaternionic contact (qc) manifold (M, g, \mathbb{Q}) is a $4n + 3$ -dimensional manifold M with a codimension three distribution H equipped with an $Sp(n)Sp(1)$ structure. Explicitly, H is the kernel of a local 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in \mathbb{R}^3 together with a compatible Riemannian metric g and a rank-three bundle \mathbb{Q} consisting of endomorphisms of H locally generated by three almost complex structures I_1, I_2, I_3 on H satisfying the identities of the imaginary unit quaternions. Thus, we have $I_1 I_2 = -I_2 I_1 = I_3$, $I_1 I_2 I_3 = -id|_H$ which are hermitian compatible with the metric $g(I_s \cdot, I_s \cdot) = g(\cdot, \cdot)$ and the following compatibility conditions hold $2g(I_s X, Y) = d\eta_s(X, Y)$.

On a qc manifold of dimension $(4n + 3) > 7$ with a fixed metric g on H there exists a canonical connection defined in [1]. Biquard also showed that there is a unique connection ∇ with torsion T and a unique supplementary subspace V to H in TM , such that:

- i) ∇ preserves the splitting $H \oplus V$ and the $Sp(n)Sp(1)$ structure on H , i.e., $\nabla g = 0, \nabla \sigma \in \Gamma(\mathbb{Q})$ for a section $\sigma \in \Gamma(\mathbb{Q})$, and its torsion on H is given by $T(X, Y) = -[X, Y]|_V$;
- ii) for $\xi \in V$, the endomorphism $T(\xi, \cdot)|_H$ of H lies in $(sp(n) \oplus sp(1))^\perp \subset gl(4n)$;
- iii) the connection on V is induced by the natural identification φ of V with \mathbb{Q} , $\nabla \varphi = 0$.

When the dimension of M is at least eleven [1] also described the supplementary *vertical distribution* V , which is (locally) generated by the so called *Reeb vector fields* $\{\xi_1, \xi_2, \xi_3\}$ determined by

$$(2.1) \quad \eta_s(\xi_k) = \delta_{sk}, \quad (\xi_s \lrcorner d\eta_s)|_H = 0, \quad (\xi_s \lrcorner d\eta_k)|_H = -(\xi_k \lrcorner d\eta_s)|_H,$$

where \lrcorner denotes the interior multiplication.

If the dimension of M is seven Duchemin shows in [3] that if we assume, in addition, the existence of Reeb vector fields as in (2.1), then the Biquard result holds. *Henceforth, by a qc structure in dimension 7 we shall mean a qc structure satisfying (2.1).* This implies the existence of the connection with properties (i), (ii) and (iii) above.

The fundamental 2-forms ω_s of the quaternionic structure are defined by

$$2\omega_{s|H} = d\eta_{s|H}, \quad \xi \lrcorner \omega_s = 0, \quad \xi \in V.$$

The torsion restricted to H has the form $T(X, Y) = -[X, Y]_{|V} = 2 \sum_{s=1}^3 \omega_s(X, Y) \xi_s$.

2.2. Invariant decompositions. Any endomorphism Ψ of H can be decomposed with respect to the quaternionic structure (\mathbb{Q}, g) uniquely into four $Sp(n)$ -invariant parts $\Psi = \Psi^{+++} + \Psi^{+--} + \Psi^{-+-} + \Psi^{---}$, where Ψ^{+++} commutes with all three I_i , Ψ^{+--} commutes with I_1 and anti-commutes with the others two, etc. The two $Sp(n)Sp(1)$ -invariant components are given by $\Psi_{[3]} = \Psi^{+++}$, $\Psi_{[-1]} = \Psi^{+--} + \Psi^{-+-} + \Psi^{---}$. These are the projections on the eigenspaces of the Casimir operator $\Upsilon = I_1 \otimes I_1 + I_2 \otimes I_2 + I_3 \otimes I_3$, corresponding, respectively, to the eigenvalues 3 and -1 , see [2]. Note here that each of the three 2-forms ω_s belongs to the $[-1]$ -component, $\omega_s = \omega_{s[-1]}$ and constitute a basis of the Lie algebra $sp(1)$.

If $n = 1$ then the space of symmetric endomorphisms commuting with all I_s is 1-dimensional, i.e., the $[3]$ -component of any symmetric endomorphism Ψ on H is proportional to the identity, $\Psi_{[3]} = -\frac{tr\Psi}{4} Id_{|H}$.

2.3. The torsion tensor. The torsion endomorphism $T_\xi = T(\xi, \cdot) : H \rightarrow H$, $\xi \in V$ will be decomposed into its symmetric part T_ξ^0 and skew-symmetric part b_ξ , $T_\xi = T_\xi^0 + b_\xi$. Biquard showed in [1] that the torsion T_ξ is completely trace-free, $tr T_\xi = tr T_\xi \circ I_s = 0$, its symmetric part has the properties $T_{\xi_i}^0 I_i = -I_i T_{\xi_i}^0$, $I_2(T_{\xi_2}^0)^{+-} = I_1(T_{\xi_1}^0)^{-+-}$, $I_3(T_{\xi_3}^0)^{-+-} = I_2(T_{\xi_2}^0)^{--}$, $I_1(T_{\xi_1}^0)^{--} = I_3(T_{\xi_3}^0)^{+-}$. The skew-symmetric part can be represented as $b_{\xi_i} = I_i U$, where U is a traceless symmetric $(1,1)$ -tensor on H which commutes with I_1, I_2, I_3 . Therefore we have $T_{\xi_i} = T_{\xi_i}^0 + I_i U$. When $n = 1$ the tensor U vanishes identically, $U = 0$, and the torsion is a symmetric tensor, $T_\xi = T_\xi^0$.

The two $Sp(n)Sp(1)$ -invariant trace-free symmetric 2-tensors on H

$$(2.2) \quad T^0(X, Y) = g((T_{\xi_1}^0 I_1 + T_{\xi_2}^0 I_2 + T_{\xi_3}^0 I_3)X, Y) \quad \text{and} \quad U(X, Y) = g(uX, Y)$$

were introduced in [4] and enjoy the properties

$$(2.3) \quad \begin{aligned} T^0(X, Y) + T^0(I_1 X, I_1 Y) + T^0(I_2 X, I_2 Y) + T^0(I_3 X, I_3 Y) &= 0, \\ U(X, Y) &= U(I_1 X, I_1 Y) = U(I_2 X, I_2 Y) = U(I_3 X, I_3 Y). \end{aligned}$$

From [8, Proposition 2.3] we have

$$(2.4) \quad 4T^0(\xi_s, I_s X, Y) = T^0(X, Y) - T^0(I_s X, I_s Y),$$

hence, taking into account (2.4) it follows

$$(2.5) \quad \begin{aligned} T(\xi_s, I_s X, Y) &= T^0(\xi_s, I_s X, Y) + g(I_s u I_s X, Y) \\ &= \frac{1}{4} [T^0(X, Y) - T^0(I_s X, I_s Y)] - U(X, Y). \end{aligned}$$

2.4. Torsion and curvature. Let $R = [\nabla, \nabla] - \nabla_{[\cdot, \cdot]}$ be the curvature tensor of ∇ and the dimension is $4n + 3$. We denote the curvature tensor of type (0,4) and the torsion tensor of type (0,3) by the same letter, $R(A, B, C, D) := g(R(A, B)C, D)$, $T(A, B, C) := g(T(A, B), C)$, $A, B, C, D \in \Gamma(TM)$. The *qc-Ricci tensor* Ric , *normalized qc-scalar curvature* S of the Biquard connection are defined, respectively, by the following formulas (cf. Convention 1.3), $Ric(A, B) = \sum_{b=1}^{4n} R(e_b, A, B, e_b)$, $8n(n+2)S = \sum_{a,b=1}^{4n} R(e_b, e_a, e_a, e_b)$. The qc-Ricci tensor and the normalized qc-scalar curvature are determined by the torsion of the Biquard connection as follows [4]

$$(2.6) \quad \begin{aligned} Ric(X, Y) &= (2n+2)T^0(X, Y) + (4n+10)U(X, Y) + 2(n+2)Sg(X, Y), \\ T(\xi_i, \xi_j) &= -S\xi_k - [\xi_i, \xi_j]_H, \quad S = -g(T(\xi_1, \xi_2), \xi_3). \end{aligned}$$

Note that for $n = 1$ the above formulas hold with $U = 0$.

Any 3-Sasakian manifold has zero torsion endomorphism, and the converse is true if in addition the qc-scalar curvature is a positive constant [4].

2.5. The Ricci identities. We use repeatedly the Ricci identities of order two and three, see also [8]. Let f be a smooth function on the qc manifold M with horizontal gradient ∇f defined by $g(\nabla f, X) = df(X)$. The sub-Laplacian of f is $\Delta f = -\sum_{a=1}^{4n} \nabla^2 f(e_a, e_a)$. We have the following Ricci identities (see e.g. [4, 9])

$$\begin{aligned} \nabla^2 f(X, Y) - \nabla^2 f(Y, X) &= -2 \sum_{s=1}^3 \omega_s(X, Y) df(\xi_s), \\ \nabla^2 f(X, \xi_s) - \nabla^2 f(\xi_s, X) &= T(\xi_s, X, \nabla f), \\ \nabla^3 f(X, Y, Z) - \nabla^3 f(Y, X, Z) &= -R(X, Y, Z, \nabla f) - 2 \sum_{s=1}^3 \omega_s(X, Y) \nabla^2 f(\xi_s, Z). \end{aligned}$$

We also need the qc-Bochner formula [5, (4.1)]

$$(2.7) \quad \begin{aligned} \frac{1}{2} \Delta |\nabla f|^2 &= |\nabla^2 f|^2 - g(\nabla(\Delta f), \nabla f) + 2(n+2)S|\nabla f|^2 + 2(n+2)T^0(\nabla f, \nabla f) \\ &\quad + 2(2n+2)U(\nabla f, \nabla f) + 4 \sum_{s=1}^3 \nabla^2 f(\xi_s, I_s \nabla f). \end{aligned}$$

2.6. The horizontal divergence theorem. Let (M, g, \mathbb{Q}) be a qc manifold of dimension $4n+3 \geq 7$. For a fixed local 1-form η and a fixed $s \in \{1, 2, 3\}$ the form $Vol_\eta = \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \omega_s^{2n}$ is a locally defined volume form. Note that Vol_η is independent of s as well as the local one forms η_1, η_2, η_3 . Hence, it is a globally defined volume form. The (horizontal) divergence of a horizontal vector field/one-form $\sigma \in \Lambda^1(H)$, defined by $\nabla^* \sigma = -tr|_H \nabla \sigma = -\nabla \sigma(e_a, e_a)$ supplies the integration by parts formula, [4], see also [12],

$$\int_M (\nabla^* \sigma) Vol_\eta = 0.$$

2.7. The P -form. We recall the definition of the P -form from [6]. Let (M, g, \mathbb{Q}) be a compact quaternionic contact manifold of dimension $4n + 3$ and f a smooth function on M .

For a smooth function f on M the P -form $P \equiv P_f \equiv P[f]$ on M is defined by [6]

$$(2.8) \quad \begin{aligned} P_f(X) = & \nabla^3 f(X, e_b, e_b) + \sum_{t=1}^3 \nabla^3 f(I_t X, e_b, I_t e_b) - 4n Sdf(X) + 4n T^0(X, \nabla f) \\ & - \frac{8n(n-2)}{n-1} U(X, \nabla f), \quad \text{if } n > 1, \\ P_f(X) = & \nabla^3 f(X, e_b, e_b) + \sum_{t=1}^3 \nabla^3 f(I_t X, e_b, I_t e_b) - 4Sdf(X) + 4T^0(X, \nabla f), \quad \text{if } n = 1. \end{aligned}$$

The C -operator is the fourth-order differential operator independent of f defined by

$$Cf = -\nabla^* P_f = (\nabla_{e_a} P_f)(e_a).$$

We say that the P -function of f is non-negative if its integral exists and is non-positive

$$(2.9) \quad \int_M f \cdot Cf \, Vol_\eta = - \int_M P_f(\nabla f) \, Vol_\eta \geq 0.$$

If (2.9) holds for any smooth function of compact support we say that the C -operator is non-negative. It turns out that the C -operator is non-negative on any compact qc manifold of dimension at least eleven [6].

One of the key identities which relates the P -function and the qc Bochner formula (2.7) on a compact manifolds is the next identity, (dropping the last term when $n = 1$), [6, (3.4)]

$$(2.10) \quad \begin{aligned} & \int_M \sum_{s=1}^3 \nabla^2 f(\xi_s, I_s \nabla f) \, Vol_\eta \\ & = \int_M \left[-\frac{1}{4n} P_f(\nabla f) - \frac{1}{4n} (\Delta f)^2 - S|\nabla f|^2 + \frac{(n+1)}{n-1} U(\nabla f, \nabla f) \right] Vol_\eta. \end{aligned}$$

3. THE QC HEAT EQUATION AND ITS ENERGY FUNCTIONAL

The next lemma is crucial for the proof of our main result.

Lemma 3.1. *Let (M, g, \mathbb{Q}) be a compact $4n + 3$ -dimensional quaternionic contact manifold. Then the next formula holds*

$$(3.1) \quad \begin{aligned} \alpha^2 \frac{d}{dt} \mathcal{F}(\varphi) = & \frac{4\alpha}{3(1-2\alpha)} \int_M F^{\frac{1}{\alpha}-2} (\Delta F)^2 \, Vol_\eta \\ & + \frac{48n\alpha^2 - 2(16n-3)\alpha - 3}{12(2n+1)\alpha^2} \int_M F^{\frac{1}{\alpha}-4} |\nabla F|^4 \, Vol_\eta + \frac{4(3-4\alpha)\alpha^2}{(2n+1)(1-2\alpha)} \int_M P_{F^{\frac{1}{2\alpha}}} (\nabla F^{\frac{1}{2\alpha}}) \, Vol_\eta \\ & - \frac{2n(3-4\alpha)}{3(n+2)(1-2\alpha)} \int_M F^{\frac{1}{\alpha}-2} L(\nabla F, \nabla F) \, Vol_\eta - \frac{4n(3-4\alpha)}{3(2n+1)(1-2\alpha)} \int_M F^{\frac{1}{\alpha}-2} p(F) \, Vol_\eta. \end{aligned}$$

In the formula (3.1), $P_{F^{\frac{1}{2\alpha}}}(\nabla F^{\frac{1}{2\alpha}})$ is the P -function defined in (2.8) of $F^{\frac{1}{2\alpha}}$, $L(\nabla F, \nabla F)$ is the left-hand side of the Lichnerowicz' type assumption (1.3) with $X := \nabla F$ and

$$p(F) \stackrel{def}{=} |\nabla^2 F|^2 - \frac{1}{4n}(\Delta F)^2 - \frac{1}{4n} \sum_{s=1}^3 [g(\nabla^2 F, \omega_s)]^2$$

is a non-negative function on M .

3.1. Proof of Lemma 3.1. The next relation between the sub-Laplacians of u and φ holds

$$(3.2) \quad \Delta u = -\frac{\Delta \varphi + |\nabla \varphi|^2}{e^\varphi},$$

which follows easily by the definitions of Δ and φ . We get the formula

$$(3.3) \quad \frac{\partial}{\partial t} \varphi = -\Delta \varphi - |\nabla \varphi|^2,$$

as a simply consequence of the definition of φ , (1.1) and (3.2). Further, the next chain of equalities holds

$$(3.4) \quad \begin{aligned} \frac{d}{dt} \mathcal{F}(\varphi) &= \frac{d}{dt} \int_M \left(-\Delta \varphi - \frac{\partial}{\partial t} \varphi \right) u \, Vol_\eta = -\frac{d}{dt} \int_M \Delta \varphi u \, Vol_\eta + \frac{d}{dt} \int_M \left(\frac{\partial}{\partial t} u \right) Vol_\eta \\ &= -\int_M \left[\left(\Delta \frac{\partial}{\partial t} \varphi \right) u + \Delta \varphi \frac{\partial}{\partial t} u \right] Vol_\eta = -\int_M \left(\frac{\partial}{\partial t} \varphi - \Delta \varphi \right) \Delta u \, Vol_\eta \\ &= \int_M e^{-\varphi} \left[-2(\Delta \varphi)^2 - 3\Delta \varphi |\nabla \varphi|^2 - |\nabla \varphi|^4 \right] Vol_\eta, \end{aligned}$$

where we used (3.3) for the first equality, the definition of φ for the second one, (1.1) and the divergence theorem for the third equality. Finally, we took into account the self-adjointness of the sub-Laplacian to obtain the fourth equality and (3.2), (3.3) for the last one.

We need the next two identities:

$$(3.5) \quad |\nabla \varphi|^2 = \alpha^{-2} F^{-2} |\nabla F|^2, \quad \Delta \varphi = -\alpha^{-1} \left(F^{-2} |\nabla F|^2 + F^{-1} \Delta F \right),$$

which, substituted into (3.4), give

$$(3.6) \quad \begin{aligned} \alpha^2 \frac{d}{dt} \mathcal{F}(\varphi) &= -2 \int_M F^{\frac{1}{\alpha}-2} (\Delta F)^2 Vol_\eta \\ &\quad + (3 - 4\alpha) \alpha^{-1} \int_M F^{\frac{1}{\alpha}-3} \Delta F |\nabla F|^2 Vol_\eta + (-1 + 3\alpha - 2\alpha^2) \alpha^{-2} \int_M F^{\frac{1}{\alpha}-4} |\nabla F|^4 Vol_\eta. \end{aligned}$$

Next, we consider the (horizontal) vector field $F^{\frac{1}{\alpha}-2} |\nabla F|^2$, in order to deal with the term $\int_M F^{\frac{1}{\alpha}-3} \Delta F |\nabla F|^2 Vol_\eta$ in (3.6). We get by some standard calculations, using the divergence formula,

$$(3.7) \quad \begin{aligned} 0 &= -\int_M \nabla^* \left(F^{\frac{1}{\alpha}-2} |\nabla F|^2 \right) Vol_\eta \\ &= \int_M g \left(\nabla (F^{\frac{1}{\alpha}-2} \Delta F), \nabla F \right) Vol_\eta - \int_M F^{\frac{1}{\alpha}-2} \Delta F \nabla^* \nabla F Vol_\eta \\ &= \int_M F^{\frac{1}{\alpha}-2} g \left(\nabla (\Delta F), \nabla F \right) Vol_\eta + \left(\frac{1}{\alpha} - 2 \right) \int_M F^{\frac{1}{\alpha}-3} \Delta F |\nabla F|^2 Vol_\eta - \int_M F^{\frac{1}{\alpha}-2} (\Delta F)^2 Vol_\eta. \end{aligned}$$

Integrate the qc-Bochner formula (2.7) over the compact M and use (3.7) to get

$$\begin{aligned}
 (3.8) \quad & \left(\frac{1}{\alpha} - 2\right) \int_M F^{\frac{1}{\alpha}-3} \Delta F |\nabla F|^2 \text{Vol}_\eta \\
 &= \int_M F^{\frac{1}{\alpha}-2} \left[-\frac{1}{2} \Delta |\nabla F|^2 - |\nabla^2 F|^2 - 2(n+2)S |\nabla F|^2 - 2(n+2)T^0(\nabla F, \nabla F) \right. \\
 &\quad \left. - 2(2n+2)U(\nabla F, \nabla F) - 4 \sum_{s=1}^3 \nabla^2 F(\xi_s, I_s \nabla F) + (\Delta F)^2 \right] \text{Vol}_\eta.
 \end{aligned}$$

The next step is to find some suitable representations of the two terms $\int_M F^{\frac{1}{\alpha}-2} \Delta |\nabla F|^2 \text{Vol}_\eta$ and $\int_M F^{\frac{1}{\alpha}-2} \sum_{s=1}^3 \nabla^2 F(\xi_s, I_s \nabla F) \text{Vol}_\eta$. To deal with the first, we consider the (horizontal) vector field $F^{\frac{1}{\alpha}-2} \nabla |\nabla F|^2$. We obtain the next sequence of equalities, using the divergence formula and some standard calculations:

$$\begin{aligned}
 (3.9) \quad 0 &= - \int_M \nabla^* \left(F^{\frac{1}{\alpha}-2} \nabla |\nabla F|^2 \right) \text{Vol}_\eta \\
 &= \left(\frac{1}{\alpha} - 2\right) \int_M F^{\frac{1}{\alpha}-3} g(\nabla F, \nabla |\nabla F|^2) \text{Vol}_\eta - \int_M F^{\frac{1}{\alpha}-2} \Delta |\nabla F|^2 \text{Vol}_\eta \\
 &= \left(\frac{1}{\alpha} - 2\right) \int_M F^{\frac{1}{\alpha}-3} |\nabla F|^2 \Delta F \text{Vol}_\eta - \left(\frac{1}{\alpha} - 2\right) \left(\frac{1}{\alpha} - 3\right) \int_M F^{\frac{1}{\alpha}-4} |\nabla F|^4 \text{Vol}_\eta \\
 &\quad - \int_M F^{\frac{1}{\alpha}-2} \Delta |\nabla F|^2 \text{Vol}_\eta.
 \end{aligned}$$

To get the third equality in (3.9) we used the identity

$$\begin{aligned}
 0 &= \int_M \nabla^* \left(F^{\frac{1}{\alpha}-3} |\nabla F|^2 \nabla F \right) \text{Vol}_\eta = - \int_M F^{\frac{1}{\alpha}-3} |\nabla F|^2 \Delta F \text{Vol}_\eta \\
 &\quad + \int_M F^{\frac{1}{\alpha}-3} g(\nabla F, \nabla |\nabla F|^2) \text{Vol}_\eta + \left(\frac{1}{\alpha} - 3\right) \int_M F^{\frac{1}{\alpha}-4} |\nabla F|^4 \text{Vol}_\eta
 \end{aligned}$$

in order to take an appropriate representation of the term $\int_M F^{\frac{1}{\alpha}-3} g(\nabla F, \nabla |\nabla F|^2) \text{Vol}_\eta$.

To handle the term $\int_M F^{\frac{1}{\alpha}-2} \sum_{s=1}^3 \nabla^2 F(\xi_s, I_s \nabla F) \text{Vol}_\eta$ we use the next formula [5, (3.12)]

$$(3.10) \quad \int_M \sum_{s=1}^3 \nabla^2 f(\xi_s, I_s \nabla f) \text{Vol}_\eta = - \int_M \left[4n \sum_{s=1}^3 (df(\xi_s))^2 + \sum_{s=1}^3 T(\xi_s, I_s \nabla f, \nabla f) \right] \text{Vol}_\eta.$$

Set $f := F^{\frac{1}{2\alpha}}$ into (3.10) to get after some calculations that

$$\begin{aligned}
 (3.11) \quad & \int_M F^{\frac{1}{\alpha}-2} \sum_{s=1}^3 \nabla^2 F(\xi_s, I_s \nabla F) \text{Vol}_\eta \\
 &= - \int_M F^{\frac{1}{\alpha}-2} \left[4n \sum_{s=1}^3 \left(dF(\xi_s) \right)^2 + \sum_{s=1}^3 T(\xi_s, I_s \nabla F, \nabla F) \right] \text{Vol}_\eta
 \end{aligned}$$

Now, we substitute (3.9), (3.11) in (3.8) and use the properties of the torsion tensor (2.3), (2.5) to obtain the identity

$$\begin{aligned}
(3.12) \quad & \frac{3}{2} \left(\frac{1}{\alpha} - 2 \right) \int_M F^{\frac{1}{\alpha}-3} \Delta F |\nabla F|^2 Vol_\eta = \frac{1}{2} \left(\frac{1}{\alpha} - 2 \right) \left(\frac{1}{\alpha} - 3 \right) \int_M F^{\frac{1}{\alpha}-4} |\nabla F|^4 Vol_\eta \\
& - \int_M F^{\frac{1}{\alpha}-2} \left[|\nabla^2 F|^2 + 2(n+2)S|\nabla F|^2 + 2nT^0(\nabla F, \nabla F) + 4(n+4)U(\nabla F, \nabla F) \right. \\
& \quad \left. - 16n \sum_{s=1}^3 \left(dF(\xi_s) \right)^2 - (\Delta F)^2 \right] Vol_\eta.
\end{aligned}$$

Substitute the right-hand side of (3.10) into (2.10) one obtains for $f := F^{\frac{1}{2\alpha}}$ the formula

$$\begin{aligned}
(3.13) \quad & -4n \int_M F^{\frac{1}{\alpha}-2} \sum_{s=1}^3 \left(dF(\xi_s) \right)^2 Vol_\eta \\
& = \int_M \left[-\frac{\alpha^2}{n} P_{F^{\frac{1}{2\alpha}}}(\nabla F^{\frac{1}{2\alpha}}) - \frac{1}{4n} F^{\frac{1}{\alpha}-2} (\Delta F)^2 + \frac{1}{2n} \left(\frac{1}{2\alpha} - 1 \right) F^{\frac{1}{\alpha}-3} \Delta F |\nabla F|^2 \right. \\
& \quad \left. - \frac{1}{4n} \left(\frac{1}{2\alpha} - 1 \right)^2 F^{\frac{1}{\alpha}-4} |\nabla F|^4 - F^{\frac{1}{\alpha}-2} \left(S|\nabla F|^2 - T^0(\nabla F, \nabla F) + \frac{2(n-2)}{n-1} U(\nabla F, \nabla F) \right) \right] Vol_\eta.
\end{aligned}$$

It follows from the inequalities [5, (4.6), (4.7)] the next representation of the norm of the horizontal Hessian:

$$(3.14) \quad |\nabla^2 F|^2 = \frac{1}{4n} (\Delta F)^2 + \frac{1}{4n} \sum_{s=1}^3 [g(\nabla^2 F, \omega_s)]^2 + p(F),$$

where $p(F)$ is a non-negative function on M .

Now, a substitution of (3.13) and (3.14) in (3.12) give the identity

$$\begin{aligned}
(3.15) \quad & \int_M F^{\frac{1}{\alpha}-3} \Delta F |\nabla F|^2 Vol_\eta = \frac{8\alpha^3}{(3n+2)(1-2\alpha)} \int_M P_{F^{\frac{1}{2\alpha}}}(\nabla F^{\frac{1}{2\alpha}}) Vol_\eta \\
& + \frac{2n+1-2(3n+1)\alpha}{2(3n+2)\alpha} \int_M F^{\frac{1}{\alpha}-4} |\nabla F|^4 Vol_\eta + \frac{(3+4n)\alpha}{2(3n+2)(1-2\alpha)} \int_M F^{\frac{1}{\alpha}-2} (\Delta F)^2 Vol_\eta \\
& - \frac{2n\alpha}{(3n+2)(1-2\alpha)} \int_M F^{\frac{1}{\alpha}-2} \left[2nS|\nabla F|^2 + 2(n+2)T^0(\nabla F, \nabla F) + \frac{4n(n+1)}{n-1} U(\nabla F, \nabla F) \right] Vol_\eta \\
& - \frac{2n\alpha}{(3n+2)(1-2\alpha)} \int_M F^{\frac{1}{\alpha}-2} \left[\frac{1}{4n} \sum_{s=1}^3 [g(\nabla^2 F, \omega_s)]^2 + p(F) \right] Vol_\eta.
\end{aligned}$$

Note that we have the representation

$$\begin{aligned}
(3.16) \quad & 2nS|\nabla F|^2 + 2(n+2)T^0(\nabla F, \nabla F) + \frac{4n(n+1)}{n-1} U(\nabla F, \nabla F) \\
& = -S|\nabla F|^2 + T^0(\nabla F, \nabla F) - \frac{2(n-2)}{n-1} U(\nabla F, \nabla F) + \frac{2n+1}{2(n+2)} L(\nabla F, \nabla F).
\end{aligned}$$

Moreover, we obtain from the formula [7, (4.12)]

$$\int_M \left[-S|\nabla f|^2 + T^0(\nabla f, \nabla f) - \frac{2(n-2)}{n-1} U(\nabla f, \nabla f) \right] Vol_\eta = \int_M \left[\frac{1}{4n} P_f(\nabla f) + \frac{1}{4n} (\Delta f)^2 \right] Vol_\eta$$

$$- \frac{1}{4n} \sum_{s=1}^3 [g(\nabla^2 f, \omega_s)]^2 \Big] Vol_\eta$$

with $f := F^{\frac{1}{2\alpha}}$ the next identity:

$$(3.17) \quad \int_M F^{\frac{1}{\alpha}-2} \left[-S|\nabla F|^2 + T^0(\nabla F, \nabla F) - \frac{2(n-2)}{n-1} U(\nabla F, \nabla F) \right] Vol_\eta \\ = \int_M \left\{ \frac{1}{4n} \left[F^{\frac{1}{\alpha}-2} (\Delta F)^2 - 2 \left(\frac{1}{2\alpha} - 1 \right) F^{\frac{1}{\alpha}-3} \Delta F |\nabla F|^2 + \left(\frac{1}{2\alpha} - 1 \right)^2 F^{\frac{1}{\alpha}-4} |\nabla F|^4 \right] \right. \\ \left. + \frac{\alpha^2}{n} P_{F^{\frac{1}{2\alpha}}}(\nabla F^{\frac{1}{2\alpha}}) - \frac{1}{4n} F^{\frac{1}{\alpha}-2} \sum_{s=1}^3 [g(\nabla^2 F, \omega_s)]^2 \right\} Vol_\eta.$$

Taking into account (3.16) and (3.17) in (3.15), we get after some simple calculations

$$(3.18) \quad \frac{3(2n+1)}{2} \int_M F^{\frac{1}{\alpha}-3} \Delta F |\nabla F|^2 Vol_\eta = \frac{8n+3-6(4n+1)\alpha}{8\alpha} \int_M F^{\frac{1}{\alpha}-4} |\nabla F|^4 Vol_\eta \\ + \frac{(2n+1)\alpha}{1-2\alpha} \int_M F^{\frac{1}{\alpha}-2} (\Delta F)^2 Vol_\eta + \frac{6\alpha^3}{1-2\alpha} \int_M P_{F^{\frac{1}{2\alpha}}}(\nabla F^{\frac{1}{2\alpha}}) Vol_\eta - \frac{2n\alpha}{1-2\alpha} \int_M F^{\frac{1}{\alpha}-2} p(F) Vol_\eta \\ - \frac{n(2n+1)\alpha}{(n+2)(1-2\alpha)} \int_M F^{\frac{1}{\alpha}-2} L(\nabla F, \nabla F) Vol_\eta,$$

which is the needed representation of the term $\int_M F^{\frac{1}{\alpha}-3} \Delta F |\nabla F|^2 Vol_\eta$.

Finally, we substitute (3.18) into (3.6) to obtain (3.1). This ends the proof of Lemma 3.1.

3.2. Proofs of Theorem 1.1. The polynomial $h_n(\alpha) \stackrel{def}{=} 48n\alpha^2 - 2(16n-3)\alpha - 3$ that appears in the right-hand side of (3.1) is non-positive for $\alpha \in [\frac{16n-3-\sqrt{256n^2+48n+9}}{48n}, \frac{16n-3+\sqrt{256n^2+48n+9}}{48n}]$. If we choose $\alpha \in [\frac{16n-3-\sqrt{256n^2+48n+9}}{48n}, 0)$ and suppose that the conditions (i) and (ii) of Theorem 1.1 hold, it is easy to see that any summand in the right-hand side of (3.1) is non-positive, which proofs Theorem 1.1.

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(Stefan Ivanov) UNIVERSITY OF SOFIA, FACULTY OF MATHEMATICS AND INFORMATICS, BLVD. JAMES BOURCHIER 5, 1164, SOFIA, BULGARIA

AND INSTITUTE OF MATHEMATICS AND INFORMATICS, BULGARIAN ACADEMY OF SCIENCES
E-mail address: `ivanovsp@fmi.uni-sofia.bg`

(Alexander Petkov) UNIVERSITY OF SOFIA, FACULTY OF MATHEMATICS AND INFORMATICS, BLVD. JAMES BOURCHIER 5, 1164, SOFIA, BULGARIA

E-mail address: `a_petkov_fmi@abv.bg`